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Arithmetic, One Child at a Time

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Introduction

This book is written for people who want to help a child do arithmetic with whole numbers in a healthy, confident way. Many adults have experienced severe difficulties with mathematics, themselves. They may hate or fear the subject. Even so, such a person can be the best resource available to help a child avoid problems with math, and come to enjoy it. The following strengths are adequate for the task.

First, the child needs someone who can plan long term projects using many short time periods. Children may not think about math very much until they are required to do so, in long lessons at school. But math is more fun when it is done outside of school, before it is required, as a challenge not requiring immediate success. It is stressful to do math on a schedule, under threat of criticism or embarrassment. You can help by planning to bring up ideas about math before they are needed.

Second, the child needs someone who is on his or her side, ready to love and support her whether she shows interest or ability in mathematics or not. You can help by not insisting that she do anything uncomfortable.

Third, the child needs someone who already has been through some study of mathematics, even if the experience was unpleasant. This book provides enough background to get started, and there are other books in this series that will be helpful.

The methods described in this book require the strength and energy a helper with those three capacities; they don't require an expert mathematician. A teacher responsible for twenty or more students can't be expected to find time for the style of assistance described here. Often, in a standard classroom, children have little opportunity to work on mathematics except by practicing repetitive, uninteresting exercises, perhaps without understanding their meaning or any purpose other than preparing for a test. What's needed is a loving helper for each child, sometimes for just a minute or two at a time, to inspire her or him to do just as much math as comes naturally. If there is drudgery to be done, it can be left for the schools. Drudgery is not included in this book. If we stay ahead of schools, perhaps it can be avoided altogether.

Another book in this series, Math Games, uses materials that make these concepts easier to understand and use. It would be desirable to use this book and the Math Games book together. But if you are stranded with a child, in a car or on a desert island, without game materials available, this book suggests ways to pass the time pleasantly and productively, with no more than a pencil and paper and the challenges of an active mind.

For explanations of more difficult concepts of arithmetic, you may want to read early parts of the book, Tutoring Algebra, also in this series.

How to Use This Book

1. Don’t explain things. Instead, ask questions that will help the child figure things out for himself or herself.

2. Be flexible. If the child can’t answer a question, think of an easier question. If the child is bored, think of a new concept to ask about.

3. Whenever a child wants to quit doing math with you, permit her to quit immediately. Be willing to abandon this project for days, months or years.

The dialogs in this book are only skeletons. If a lesson really proceeds as fast as an example in the book, then the child probably already understood the concept discussed. Actual lessons will usually be shorter and more numerous. But a book that displayed more realistic, meandering discussions would be less instructive.

Feel free to depart from the book. So long as you just ask questions and don’t force the child to do anything uncomfortable, you aren’t likely to do any damage. A little experience with the child will prepare you to be a better tutor, as you learn to guess which questions she might be able to answer.

In case you help a child look at this book, some parts are printed in large print that is easy to read. You may soon find that the flexibility of reading in advance and then starting with a blank piece of paper is more effective. In that case, your child will have your undivided attention as you discuss arithmetic, instead of seeing that part of your attention is on this book.

If you write as you go, you can be more flexible in following your student’s interests and answering questions that may come up. Even if your student reads the book with you, it will help to have a pencil and paper handy, just in case you think of something different to do. It is possible to make dots for counting as fast as one can tap a drum, moving the pencil to the side while tapping.

Counting



Helper: Can you count these dots?

Student: One, two, three, four, five, six. . . . .

H: Good. You count very well. Now could you count very slowly, so that I can put my finger on these dots as you count. And when we run out of dots, you can quit counting.

S: One, two, three, four. . . .

H: Stop. That's good. Can you put your finger on them as you count?

S: One, two, three, four.

H: Good. Now, can you count these dots?



S: One, two, three, four, five.

H: Right. So how many dots did you count, this time?

Comments: It is harder to count things than to just count.

Our objective is for the child to understand, not to practice. Once counting dots becomes easy, it is time to find another challenge, unless the child prefers to continue with the same exercise, for entertainment. For beginners, this exercise can take weeks.

Reading Numbers

3

H: Can you read this number?

S: Three.

H: That's right.

8

H: Can you read this one?

S: (No answer.)

H: May I tell you what it is, or do you want to think about it?

S: Tell me.

H: It is an eight. Here are eight fingers. Let me see you count them.

S: One, two, three, four, five, six, seven, eight.

H: Right. Now I am going to make a copy of this eight and tape it up on the wall. Then the next time I think of it, I will ask you if you remember what it is. Do you remember right now?

S: Yes.

H: Would you say this number for me, please?

S: Eight.

H: Right. Do you think you will remember it tomorrow?

S: Yes.

Comments: Sounds of letters, shapes and colors can be taught without pressure by this method. Questions that have become boring may be taken down from the wall, but it is nice to save them and check the student's memory in a month or two, so that she can appreciate how much she has learned.

Most children need just one new memory question at a time. They usually think they will be able to remember things the next day, but aren't able to. It usually takes several days to learn to remember something new by this method, even though they may learn several new words in a day without consciously trying to do so.

If you would like to use this process to help a child prepare to read, you might be interested in the book, Phonics, One Child at a Time, available in this series.

Adding One

H: When you count, do you know what number comes after three?

S: (No answer.)

H: One, two, three. . . .

S: Four!

H: Right. And do you know what number comes after four?

S: Five.

H: Right. Now here are four fingers. If I put up one more finger, do you know how many there will be?

S: No.

H: Here are four. One, two, three, four. And one more makes – one, two, three, four. . . .

S: Five.

H: Right. So what would one more than four be?

S: (No answer.)

H: Count if you need to. Here are four. One, two, three, four. And here is one more. One, two, three, four. . . .

S: Five.

H: Right. So what would one more than four be?

S: (No answer.)

H: Count if you need to.

S: Five.

H: Right. I think you have it now. Let's make sure. What is one more than three?

S: [Thinking.] Four.

H: Right. In school they write one more than three this way.

$$3 + 1$$

That says, “Three plus one.” It means the same as one more than three. And we said three plus one is what?

S: Four.

H: Right. Here is four plus one.

$$4 + 1$$

What is four plus one?

S: Five.

H: Right. I thought you had it. In the first grade they write four plus one this way.

$$\begin{array}{r} 4 \\ +1 \\ \hline \end{array}$$

H: Then they write the answer under the line. Can you write the answer? What is the answer, again?

S: Five.

H: Right. Can you write the five under the line?

S: Yes.

$$\begin{array}{r} 4 \\ +1 \\ \hline 5 \end{array}$$

H: Good. And in high school they write it all on one line.

$$4 + 1 =$$

That last symbol says, “equals.” Would you write the five after the “equals” symbol, right here, please?

S: Why do I have to?

H: Well, you don’t have to. But it makes you smart to do this stuff. It is fun for me to see you use numbers. Is it fun for you?

S: I guess so.

H: Oh, good. So, do you want to write the answer for me?

S: Okay.

$$4 + 1 = 5$$

H: Right. And which way do you want me to write. The first grade way or the high school way?

S: The high school way.

H: Okay. And then you can write it the other way in school if the teachers want you to, and you will know how to do it both ways.

When I make my fives, I save the top for last. Then they don’t look like S’s, even if I write fast. How do you want to make your fives?

S: Like S’s.

H: Okay. That’s okay with me. If you ever start writing so fast that your fives look too much like S’s instead of like fives, you can change the way you make your fives if you want to. But it is easier to start practicing my way now than to try to change later. Then you won’t have to think about it when you write well. It will be automatic, and your brain will be free to think about what you are writing, instead of how you are writing it.

Comments: I feel encouraged when students choose the high school way of writing arithmetic, because that choice may indicate confidence. Still, I am altogether accepting when they choose the first grade way. In either case, they see that the choice is theirs.

There are sets of blocks called “base ten blocks” that are useful in helping children learn to count and add. Medium sized blocks in the set are the same size as ten small blocks, and large blocks are the same size as one hundred small blocks. This book uses dots in case the blocks aren’t available. Sometimes it is also convenient to use pennies and dimes for counting.

Adding Two

H: Do you know what two more than three is?

S: No.

H: You can start with three and count up two more numbers. One, two, three. . . .

S: Four.

H: Right. Now count up one more number. What comes after four?

S: (No answer.)

H: May I help you?

S: Yes.

H: One, two, three. . . . .

S: Four.

H: Right. Four. . . .

S: (No answer.)

H: One, two, three, four. . . .

S: Five.

H: Right. This time, just think the four and say the five. What is two more than three. One, two, three, (whispering) just think the four; don't say it. Then comes. . . .

S: Five.

H: Right. Here are three fingers, and here are two more. How many fingers is that, altogether?

S: Five?

H: That's right. Let's see if you can do another one. Here are six fingers. What is two more than six. One, two, three, four, five, six. . . . and two more. One, two, three, four, five, six, ...

S: Seven, eight.

H: Good. So what is the answer? How many fingers would there be if we added two more fingers to six fingers?

S: Eight.

H: Right. Eight is two more than six. And what is two more than five?

S: (No answer.)

H: One, two three, four, five, ...

S: Seven.

H: You've got it. Two more than five is seven. I'll write five plus two for you here and tape it on the wall, to see if you can remember what five plus two is tomorrow.

5 + 2 =

And do you know now? Two more than five is ....

S: Seven.

H: Right. Now I'll see if I can get you to forget the seven by working on another one. What is three plus two?

S: (No answer.)

H: May I help you, or are you still thinking?

S: No, ... five.

H: Right. You've got it. And now, do you remember five plus two?

S: Seven.

H: Right. I'll see if you remember it tomorrow.

Comments: This exercise goes slowly for most students at first. When students ask me not to interrupt them because they are thinking, I feel encouraged, because if they enjoy thinking they will go on learning efficiently when I am not there.

Sometimes students doing the exercise described in this dialog just remember the seven, and forget the question, 5 + 2. Once they remember that seven is the answer, then even if you ask them what one plus one is, they may answer, "seven". So, you might ask, "What number plus two was seven?"

Adding Tens

H: Can you count by tens?

S: Yes. Ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, one hundred . . . .

H: Good, that's enough. Now let me draw some dots and see if you can count them by tens.

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I have learned to draw them fast by putting ten in each row. (Covering the last row of ten dots.) Would you count them by tens with me, while I point at them?

S: Ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, one hundred.

H: Right, (uncovering the last row of ten dots.) and ten more makes one hundred ....

S: I don't know.

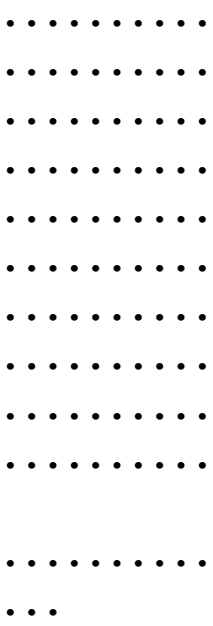
H: One hundred ten. Would you say, "One hundred ten," for me please?

S: One hundred and ten.

H: Good. In some schools they want you to say, "One hundred ten," and leave out the word, and, but I don't care how you say it. You can say "one hundred ten," or "one hundred and ten," if you want to. Either way, I will know what you mean, and so will other people.

If I put one more dot on the paper, so it is eleven more than one hundred, then the way I say it is "one hundred eleven" dots. I'll make a couple more dots more than that, so we will have thirteen more than one hundred.

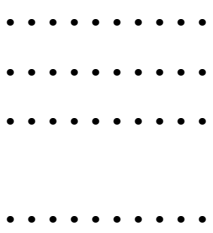
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H: Can you say the number of dots we have now?

S: One hundred thirteen.

H: Right. Now, if I had thirty dots, and drew ten more, how many would there be?



S: I don’t know.

H: You could count them by tens. Ten, twenty, thirty, ...

S: Forty.

H: Right. Forty comes after thirty, when you count by tens. And what would fifty dots plus ten more dots be?

S: (No answer.)

H: Thirty, forty, fifty, ... What comes after fifty when you count by tens?

S: Fifty-one?

H: That would be right if we were counting by ones. But counting by tens, we say, “Ten, twenty, thirty, forty, fifty, ...”

S: Sixty.

H: Right. Now let me draw the dots.

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First there are five rows with ten in each row, and then one more row. That makes six rows. And six rows makes how many dots?

S: Six?

H: No, here are six dots.

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We want to count by tens when we count those rows of dots. Ten, twenty, thirty, forty, fifty, ...

S: Sixty.

H: Right. Now let’s see if you are ready for the next question. What is forty plus ten. I’ll draw the dots.

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How many are there. Ten, twenty, ...

S: Thirty, forty, fifty.

H: Right. Now, let me see if you can draw the dots. What is twenty plus ten. Would you draw twenty dots for me?

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Good. Now leave some space and draw ten more.

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H: Thank you. It is hard to draw dots, when you have to think about each one. When I draw them, I draw like a wood pecker, very fast. You will be able to learn to do that when you get extremely ancient, like I am.

S: What does ancient mean?

H: Old. I am old, and you are young. I have had time to learn to draw dots very quickly, and my brain has had time to grow parts that let me control my body very well. Your young brain doesn't control your hands as well, but it remembers things better than my brain does.

Now, let me see – why did I ask you to draw those dots?

S: I don't know.

H: It must have been so we could figure out what twenty plus ten is. Do you know?

S: Thirty. I didn't need to draw dots to know that.

H: Oh, good. I won't ask you to draw any more dots, then. Let's see if you can do another one in your head, without drawing dots. What is eighty plus ten?

S: I don't know.

H: What comes after eighty when you count by tens?

S: Ninety.

H: Right. So what is eighty plus ten more?

S: Ninety.

H: You've got it. And what did you say twenty plus ten was?

Comments: It is important to count by tens before adding two digit numbers, so the process of adding will be intuitively meaningful, rather than just accurate. Since it is our objective for the student to understand and enjoy mathematics, it is helpful for her to think about adding dots or blocks before learning the standard addition process with written numbers. We want her to feel confident that she understands the meaning of what she is doing.

As mentioned earlier, each section in this book covers more arithmetic than a child would actually learn in one session. A child who is encountering these concepts for the first time might take weeks or months to complete this dialog. Follow the child's lead, and discuss whatever topic comes up, in the same helpful spirit, rather than insisting that there is work to be done. We want these activities to be entertaining, so rely on curiosity and social motivation, rather than diligence.

Adding Two Digit Numbers

(easy problems, such as 30 + 14)

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H: Here are twenty dots and eleven dots. Do you know how many dots there are all together?

S: (No answer.)

H: First, let me cover up the last one.

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What is twenty plus ten?

S: Thirty.

H: Right. (Uncovering the dot.) Twenty plus eleven is one more than twenty plus ten. How many would that be?

S: (No answer.)

H: When I cover the last dot, we have twenty plus ten dots. How many did you say that is?

S: Thirty.

H: Right. So when I uncover the last dot, how many are there, all together?

S: I don't know.

H: What is one more than thirty?

S: (No answer.)

H: If you count by ones, what comes after thirty? Twenty-eight, twenty-nine, thirty….

S: Thirty-one.

H: Right. So (Covering the last eleven dots.) twenty plus (Uncovering the last eleven dots.) eleven is what?

S: Thirty-one.

H: That's right. Now let me stick in another dot. What is twenty plus twelve?

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S: Thirty-two.

H: That's good. Can you tell me what twenty plus fifteen is?

S: Thirty-five.

H: Right. What is thirty plus fifteen?

S: (No answer.)

H: May I help you?

S: Yes.

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H: If I cover up the last five dots, we have thirty plus ten dots. What is thirty plus ten?

S: Forty.

H: Right. Thirty plus fifteen must be five more. How many is that?

S: Forty-five.

H: Right. Let's see if you can do it with bigger numbers. Do you know what fifty plus ten is?

S: Sixty.

H: Right. So what is fifty plus fifteen? That's five more.

S: Sixty-five.

H: Right. What is eighty plus fifteen?

S: Ninety-five.

H: Good. I guess you can do all of those. Let me check one more. What is eighty plus seventeen?

S: Ninety-seven.

H: Right. Now let's see. You know what thirty plus twenty is, don't you?

S: Sure, fifty.

H: Right. Now what is thirty plus twenty-four?

S: Fifty-four.

H: Good. Let me draw the dots and write the numbers beside them.

• • • • • • • • • •
• • • • • • • • • • **30**
• • • • • • • • • •

• • • • • • • • • •
• • • • • • • • • • **24**
• • • •

Can you read these numbers?

S: Thirty. Twenty-four.

H: Good. Where did you learn to read such big numbers?

S: Television.

H: Okay, you said thirty plus twenty-four was what?

S: Fifty-four.

H: Right. We could write the problem this way.

30
+24

Would you write the fifty-four under the line, please?

30
+24
45

That's close. The fifty comes before the four in fifty-four. We need to write the digit, five, first, right here under the two. The five means fifty when it comes before the four. Let me erase that answer for you, so you can try again.

30
+24
54

Good. Now let's write it the high school way.

30 + 24 =

Would you write the answer after the equals symbol, please?

30 + 24 = *54*

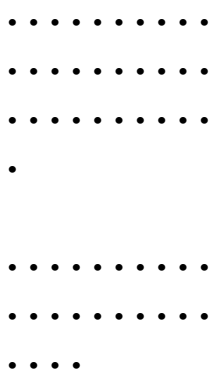
Good. Now do you want to do some harder problems, or rest up from all this thinking?

S: I want to rest.

Comments: Again, we started with dots before using written numbers, because dots are easier to understand. The spirit of thinking through problems conceptually, with dots or pennies or base ten blocks, rather than learning a faster rote procedure first, can help students enjoy math and understand it well enough to do word problems.

Some students who find it easy to figure out what three plus two equals have trouble with thirty plus twenty. In that case, it may help to ask what three plus two is, just before asking the harder question.

Adding More Two Digit Numbers



H: Can you figure out what 31 + 24 is?

S: That's too hard.

H: You could do it if I covered up the last dot in the thirty-one dots. Then we would have *thirty* plus twenty-four dots. How many would that be?

S: Fifty-four.

H: Right. Now I will uncover the other dot. How many are there now?

S: Fifty-five?

H: Good. Now, thirty-*two* plus twenty-four is one more than that. What is thirty-two plus twenty-four?

S: Fifty-six.

H: Right. What is thirty-*three* plus twenty-four?

S: I don't know.

H: Well, you know what thirty plus twenty is, right?

S: Yes, fifty.

H: And what is three plus four?

S: I don't know.

H: Do you know what three plus three is?

S: Six.

H: Right. Three plus *four* must be one more than three plus three. So three plus four is what?

S: Seven.

H: Good. Now I'll draw the dots and write the numbers.

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• • • • • • • • • • **33**
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• • • • • • • • • • **24**
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You said thirty plus twenty is what?

S: Fifty.

H: And three plus four is what?

S: Seven.

H: So how many dots are there, all together?

S: (No answer.)

H: (Covering numbers and some of the dots.)

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Here are fifty, and (uncovering dots)

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• • • • • • • • • • **33**
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. . . . . .
. . . . . **24**
. . . .

here are seven more. So all together, we have how many?

S: Fifty-seven?

H: Right. Now let me add another dot. What is thirty-*four* plus twenty-four?

. . . . . .
. . . . . **34**
. . . . . .
. . . .

. . . . . .
. . . . . **24**
. . . .

S: Fifty-eight.

H: Good. What is thirty-*six* plus twenty-four?

S: Fifty-ten.

H: Not bad. Ten more than fifty isn't really usually called fifty-ten, but I know what you mean. What is ten more than fifty?

S: I don't know.

H: If I drew the ten dots together, I bet you would know, because you could count them all by tens.

. . . . . .
. . . . . **30**
. . . . . .

. . . . . .
. . . . . **20**

. . . . . **6 + 4**

Thirty dots plus twenty dots is fifty dots. What would ten more be?

S: Sixty.

H: Right. Now, in the problem, “thirty-six plus twenty-four,” you need to think of putting the ten dots together for yourself.

• • • • • • • • • •
• • • • • • • • • • **36**
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• • • • • • • • • • **24**
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What is thirty-six plus twenty-four?

S: (No answer.)

H: May I help you?

S: Yes.

H: Here are fifty, and these two short lines of dots have ten in them. How many are there all together?

S: Sixty.

H: Good. Now let me try adding another dot to see whether you can still think through how to do these problems. What is thirty-*seven* plus twenty-four?

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• • • • • • • • • • **37**
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• • • • • • • • • • **24**
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S: Sixty-one.

H: Very good. You understand them all, now. Most people eventually learn to do these problems without drawing the dots. Here's an easy one.

31
+25

Look at the three and the two. Since they are written at the first of each number, they mean thirty and twenty. What is thirty plus twenty?

S: Fifty.

H: Good. Would you write five under the two, where it will mean fifty?

31
+25
5

Thank you. Now, what is one plus five?

S: I don't know.

H: Five plus one might be easier. What is one more than five?

S: Six.

H: Right. Do you think five plus one is the same as one plus five?

S: I don't know.

H: Let's look at the dots.

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H: First, I drew one plus five dots, and then I drew five plus one dots. Is the answer the same both times?

S: I don't know.

H: I could just move the first dot under the first five dots. Would there still be the same number of dots, if I just moved them around?

S: Yes, six.

H: Good. So do you agree that one plus five is the same a five plus one?

S: I guess.

H: You guessed right. Could you write the six under the five for me?

31
+25
56

Thank you. The five means fifty, since it is written on the left, but the six really means six, since it is written at the end of the number. Now, I will give you the hardest problem yet. Maybe you will want to work on it. You can draw the dots if you want to, or you can just use your head.

35
+48

Do you want a hint?

S: Yes.

H: Okay. It's easier if you figure out the five plus eight first, before the thirty plus forty. But if you figure out what five plus eight is, you will still have a tough problem.

Comments: Children can work on the challenging concept of adding two digit numbers and carrying, before they have memorized the addition facts. It wouldn't hurt to give them a table of addition facts to refer to, but I usually don't, since they benefit from practicing the mental process of figuring them out. You could also give them a calculator to check their answers. Most children enjoy that process.

Children usually learn to add numbers that are the same, such as 2 + 2, 3 + 3, 4 + 4, and so on, before they learn harder addition facts, so it may be easy to use these combinations to figure out the combinations of two numbers that are nearly the same, such as 3 + 4, as was done in this section.

The game, Blip, in the companion book, Math Games, helps children practice doubling numbers, and coins or base ten blocks can be used to visualize adding two digit numbers.

The answer, fifty-ten, given above, provided an opportunity for the helper to be particularly accepting. It is appropriate for the helper to understand and support every response, no matter

what it is, in order to encourage the student's initiative. Every answer, right or wrong, can be approved of, if the helper has the insight to understand the student's motivation for giving the answer.

The child in this dialog seems to have lost concentration, when he couldn't tell whether four plus one was the same as one plus four. At this point in the conversation, it might have been just as well to start talking about something that doesn't require concentration, such as giraffes or spiders.

Large Numbers

H: Can you count to one hundred?

S: Yes. One, two, ...

H: Wait a minute. I thought you could do it. I was just making sure. You don't have to actually count to one hundred.

S: But I want to.

H: Oh, well, I'll listen then. Go ahead.

S: One, two, three, ... , one hundred.

H: Great. I thought you could do it. Can you read this number?

100

S: No.

H: It says, "one hundred."

Eight tens is eighty.

80

The eight in eighty tells how many tens there are. The zero in eighty makes it ten times as big as eight.

Nine tens is ninety.

90

The nine in ninety tells how many tens there are. There are nine tens in ninety. And ten tens is one hundred. We could erase the nine in ninety and replace it with ten to make ten tens, which is one hundred.

100

It looks like a ten with one more zero. The extra zero makes it ten times as big. Can you write it?

100

Good. Thank you. Do you know what one more than one hundred is?

S: No.

H: It is one hundred one. That's pretty easy. You have answered questions like that before. What is two more than one hundred?

S: One hundred two?

H: Right. Now, I'm going to show you how to write two hundred two. You don't already know how, do you?

S: No.

H: I didn't think so. It's a little bit like writing thirty-two. Here's thirty.

30

And to write thirty-two, we could erase the last zero, and write a two.

32

You could do that, couldn't you?

S: Yes.

H: I thought so. Now, we can write one hundred two the same way. We can start with one hundred.

100

Then we can erase the last zero and write a two. Could you do it for me?

10*2*

Very good. Now, what is twenty more than one hundred?

S: One hundred twenty.

H: Right. Do you think you could write it?

S: Yes.

1020

H: That's pretty good. The problem is, to make room for the twenty, you need to erase two zeros in the one hundred. Here's one hundred.

100

Now, could you erase two zeros and write twenty?

1*20*

H: Just right. The one means one hundred when it is written beside two other digits, and then you changed those two digits to a twenty. Can you write one hundred thirty?

130

Good. Now, can you write one hundred thirty-four?

S: No.

H: First, write a one for the hundred.

1

Good. Now write the thirty-four.

134

Good. You saved time by not writing the zeros and erasing them. Now, let's see if you still remember how to write one hundred one.

11

H: Good try. You wrote a one for the hundred, and then you wrote one. But the trouble is, the number you wrote looks like eleven, because there aren't enough digits to make that first one say one hundred. Let me write one hundred,

100

And now you can erase the last zero, and replace it with a one, to make one hundred one.

10*1*

S: Is that it?

H: Yes it is. How do you feel about erasing?

S: I don't mind erasing.

H: That's good. Now, do you know what one hundred plus one hundred is?

S: One hundred one?

H: Good guess, but not right. Hundreds work like apples. Do you know what one apple plus one apple is?

S: Two, silly.

H: That is a silly question, for you, isn't it. You have known what one plus one is for a long time. And, do you mean two apples or two oranges?

S: Two.

H: One apple plus one apple is two what? Two apples or two oranges?

S: Two apples.

H: Right. And one hundred plus one hundred is two what?

S: Two hundreds.

H: Right, two hundred. Now, here's the next question. What is two apples plus one apple?

S: Three apples.

H: Right. What is two hundred plus one hundred?

S: Three hundred.

H: Right. Can you count by hundreds with me. One hundred, two hundred, three hundred, ...

S: Four hundred, five hundred, six hundred, seven hundred, eight hundred, nine hundred, ten hundred, eleven hundred, ...

H: Good; you can stop right there. Instead of ten hundred, you can say one thousand, if you want to. It means the same thing. Let me see you write three hundred

300

Good. Now, erase the three and write ten, to make ten hundred.

1000

Right. Now, I think I'll just put a comma after the one, and read the number as, "one thousand." That's a different name for ten hundred.

1,000

Can you read this next number?

3,000

S: Three thousand?

H: Right. It means the same as thirty hundred. Do you want to try to read this one?

1,300

S: Thirteen thousand?

H: That's close, but not quite right. If there were another zero at the end, it would be thirteen thousand. The one before the comma means one thousand. Now, I'll cover up the 1 and ask you to read the rest.

300

S: Three hundred.

H: Right. Now, when I uncover the one, for one thousand, the whole number is one thousand three hundred.

Now, I want you to count by hundreds with me again, and when we get to ten hundred, say, "one thousand. One hundred, two hundred, three hundred, four hundred, five hundred, six hundred, seven hundred, eight hundred, ...

S: nine hundred, one thousand.

H: Good. And one more hundred would be one thousand one hundred. And one *more* hundred would be one thousand two hundred, and then comes one thousand .....

S: Three hundred?

H: Right. Go on.

S: Four hundred.

H: Yes, one thousand four hundred. Now say one thousand five hundred.

S: One thousand five hundred,

H: Thank you. One thousand six hundred, one thousand seven hundred, …

A: One thousand eight hundred, one thousand nine hundred, one thousand ten hundred, ...

H: Right, but that ten hundred makes another thousand, so one thousand plus ten hundred would be one thousand plus another thousand. What is one thousand plus another thousand?

S: I don't know.

H: Thousands work like apples. What is one apple plus another apple?

S: Two.

H: Right. Two what. Two apples or two oranges?

S: Two apples.

H: And what do you think one thousand plus another thousand is?

S: Two.

H: Right. Two what?

S: Two thousands.

H: Right. But people say "two thousand" without the s. Now keep counting by hundreds. Two thousand one hundred, two thousand two hundred,....

S: Two thousand three hundred, two thousand four hundred, two thousand five hundred, two thousand six hundred, two thousand seven hundred, two thousand eight hundred, two thousand nine hundred, ...

H: Be careful. Ten hundred makes another thousand, so don't say, "two thousand ten hundred." What is two thousand and another thousand?

S: Two thousand one thousand?

H: What is two apples plus one apple?

S: Three apples.

H: And what is two thousand plus one thousand?

S: Three thousand.

H: Right. So when we count by hundreds, the number that comes after two thousand nine hundred is what? It's not two thousand one thousand. It is how many thousand?

S: Three thousand.

H: Right. Now, let me see. When we count by hundreds, the number that comes after, say, four thousand nine hundred would be ....

S: Five thousand.

H: Great. You've got it. Now, let's count by thousands. One thousand, two thousand, three thousand, ...

S: Four thousand, five thousand, six thousand, seven thousand, eight thousand, nine thousand, ten thousand, eleven thousand, ...

H: Good. Stop right there. Can you write ten thousand?

10000

Good. I thought you could do it. I'm going to put a comma after the number of thousands. That's what most people do.

10,000

You are getting good at using these big numbers. Would you like to count by ten thousands?

Comments: It is important to understand what these big numbers mean before learning procedures for adding and subtracting them. It is easier to count than to read numbers, and easier to read them than to write them, so the questions were asked in a good order.

I often discuss arithmetic with children when they are bored, when we are on long trips or waiting for something. Sometimes I discuss arithmetic with children I have just met, on public transportation for example, particularly if they are beginning to make trouble for their parents. Discussions about arithmetic are pleasant diversions in these settings. Children usually appreciate the attention, and take satisfaction in answering thoughtful questions. Large numbers are a particularly entertaining topic, when first encountered.

The ideas in this section can be discussed in any setting, with only pencil and paper. But base ten blocks are useful in understanding numbers that have two or three digits. It is also sometimes handy to discuss these numbers using pennies, dimes, and dollar bills. I say that a dollar bill is worth one hundred pennies, and a ten dollar bill is worth one thousand pennies.

Multiplication Facts

H: You know what ten plus ten is don't you?

S: That's easy – twenty.

H: Did you know that two times ten is the same thing?

S: No.

H: Well, it is. To get two times ten, you add up two tens. So two times ten is what?

S: Twenty.

H: In high school, we write two times ten this way.

2×10

And what is three times ten?

3×10

S: Is that three tens?

H: Yes, it is. And what would you get if you added three tens together?

S: Thirty.

H: Right. Now let me write three times five.

3×5

Can you tell me what three times five is?

S: Fifteen?

H: Right. How did you know?

S: I added up three fives.

H: Good. If we didn't know the answer, we could make three sets of five dots each and count them. Can you count by fives?

• • • • •
• • • • •
• • • • •

S: Five, ten, fifteen.

H: Good. Is that the way you did it when I asked you before?

S: Yes.

H: I thought so. Now, would five threes be the same as three fives? I'll draw the dots.

• • •
• • •
• • • **3 + 3 + 3 + 3 + 3**
• • •
• • •

• • • • •
• • • • • **5 + 5 + 5**
• • • • •

What do you think? Are there the same number of dots in both big groups?

S: Yes.

H: Good. Did you count one group from top to bottom, and the other from side to side?

S: No.

H: How did you do it?

S: I guessed.

H: Well, you guessed right. I'll show you how I know. If I turn the paper with three groups of five dots on it's side, then I get five groups of three dots. So, is five times three the same as three times five?

S: Yes.

H: And what is three times five, again?

S: Fifteen.

H: Right. Do you know what three nickles are worth?

S: Fifteen cents.

H: Right. That is probably easy for you to remember. I'm going to write those problems on a piece of paper anyway, to remind myself tomorrow to ask you whether you remember.

3 × 5

5 × 3

Maybe I'll ask, "What's three times five," or maybe I'll ask, "What's five times three?" Do you think you'll remember?

S: Sure, fifteen.

H: Right. I'll just put this little piece of paper in my pocket, to remind me to ask you.

Comments: If a child wants to learn all the multiplication facts up to nine times nine, your planning can make the project pleasant. Keep track of which facts she needs to learn, and discuss them one at a time, for just a minute or two each day. As with the rest of arithmetic, the method described here is to offer challenging play, rather than to insist on work.

By the time your child gets to multiplication, there are only twenty-one facts she is likely to need to learn. She will immediately know that one times any digit is that digit, and if she knows addition facts, she knows what two times any digit is. If she also knows that the order doesn't matter in multiplication, then after she learns what three times five is, she will know what five times three is.

Once she learns to count by fives, she will soon know all multiples of five. For example, she will know that five nickels are worth twenty-five cents. Now, if you count the multiplication facts that she still needs to learn, you will find there are no more than twenty-one of them.

If you covered one multiplication fact a day, she would see them all in less than a month, but that is too fast for most people. She is likely to forget most of them if you go that fast, and you will need to go through them again. It is probably more efficient to spend a year on the multiplication facts. If you get done with them in a year, that's fast progress. Learning one fact per week or even one fact per month is fast enough.

It took me fifteen years to memorize all the multiplication facts up to nine, because I learned to figure them out so quickly that I didn't need to memorize them. For example, two times seven is fourteen, so four times seven is twice as much. And five times eight is forty, so six times eight is eight more than forty. Simple devices such as these can replace memorization in most cases. Let your student decide whether to memorize facts or figure them out as she goes.

Before learning to multiply two digit numbers, it would help to play the game of Blip, described in the booklet, Math Games in this series, in order to practice doubling numbers. Subtraction of numbers with two or more digits is sophisticated enough that it might be better to use base ten blocks or coins when first discussing such problems, rather than drawing dots and covering some of them. When introducing subtraction, you can proceed by a method similar to the one we used for

introducing addition, with questions such as "do you know what one less than seven is," and "When you count, what number comes right before seven?"

If you want to subtract seven from forty-three, you could draw forty-three dots and cover seven of them. Or you could use four dimes and three pennies to illustrate forty-three cents. Then you could trade one of the dimes for ten pennies, so that it would be convenient to take away seven cents. This process would prepare students to understand the standard process of borrowing one. The more conceptual process, using dots or coins, needs to be done first.

Now you can probably figure out how to help with subtraction on your own. I will proceed to describe ways to introduce the concept of division.

Division

H: Do you know how many threes there are in twelve?

S: No.

H: Let me draw twelve dots.

• • • • • • • • • • • •

H: Now I'll separate three of them with a line.

• • •|• • • • • • • • •

Would you draw a line to separate three more of them for me?

• • •|• • •|• • • • • •

Thank you. Now, I'll draw another line to separate three more.

• • •|• • •|• • •|• • •

So how many groups of three are there?

S: Three.

H: Well, there are three lines. But I see one group of three dots here. How many groups are there like that?

S: Four

H: Right. Here is the way they write that in grade school.

3) $\overline{12}$

The top line is supposed to meet the curved line, but it is hard to get my computer to do that. How many threes did we say there are in twelve?

S: Four.

H: That's right. We write the four on top of the line.

4
3) 12

H: Most people say, twelve divided by three is four, because if we divide twelve into groups of three, we get four of those little groups.

H: It can also be written this way.

12 ÷ 3 =

Would you write the answer at the end for me, right after the equals sign?

S: Why?

H: Just to make sure you know where it goes. You don't have to write it if you don't want to.

S: I don't want to.

H: But you know where it goes?

S: (Pointing.) Yes, right here.

H: Good enough. Now you can read high school books if you want to. Do you want to keep talking about this arithmetic, or are you tired.

S: I'm not tired.

H: Okay, then, if you know that three times four is twelve, you must know four times three is twelve, right?

S: Right.

H: So what is twelve divided by four?

S: I don't know.

H: Do you know how many fours there in twelve?

S: Three.

H: Right. That was another way to ask the same question. What is twelve divided by four?

S: Three.

H: Right. And here's another way to ask. Four times *what* is twelve?

S: Three.

H: Right. And here is another way. What is four into twelve?

S: Three?

H: Right. Let's write that the grade school way. I'll write the "divided by" symbol the grade school way.

$$\overline{}$$
$$)$$

When we write twelve divided by four, where does the twelve go, and where does the four go?

$$4\ )\overline{12}$$

Right. I said twelve first, when I said "twelve divided by four," but you wrote four first. Maybe that's why people say, "four into twelve," instead of "twelve divided by four." Those two ways of saying it mean the same thing. Now I'll write the answer on top. What is four into twelve?

S: Three.

H: Okay. Here it is.

$$\begin{array}{r} \mathbf{3} \\ 4\ )\overline{12} \end{array}$$

Just to make sure, I'll write four times three underneath. What is four times three?

S: Twelve.

H: Okay. I'll write the twelve underneath, and draw a line, and subtract.

$$\begin{array}{r} \mathbf{3} \\ 4\ )\overline{12} \\ \mathbf{12} \\ \mathbf{0} \end{array}$$

I got zero, of course.

S: Why do you have to do that?

H: You don't really have to. If you don't bother, it is called short division, but if you write the twelve and subtract, it is called long division.

S: Long division is silly.

H: Yes, it is, for these easy problems. But if I asked how many fours there are in thirteen, instead of twelve, it might not be so easy.

S: There are still three fours in thirteen.

H: That's true. Let me write the long division.

```
   3
4 )13
  12
   1
```

This time when I subtracted, I saw that there is one left over. We could say it remains, after the three fours are subtracted, so the one is called a remainder in grade school.

S: What do they call it in high school?

H: They still call it a remainder. But sometimes, they say that one is a fourth of a four. How many fours are there in one?

S: None.

H: Right. There are no whole fours in one. But in high school, they say there is one-fourth of a four in one, and they write that one-fourth in the answer. So in high school, they would say thirteen divided by four is three and one-fourth.

```
   3 ¼
4 )13
  12
   1
```

S: I'd rather do it the grade school way.

H: Sounds good to me.

Comments. This lesson is not likely to proceed nearly this fast with a real student. Even our hypothetical student seems to be feeling a little overwhelmed by arithmetic, as students often are when division comes up. The process is often mysterious, when taught in all its grueling detail in school. It will be very helpful to introduce it in advance, in this simple setting.

The student in this dialog prefers to avoid the concept of fractions. We might check with such a student in a day or two to see if she is ready to think about them. If not, we might cut a pie into four equal pieces, and then refer to each piece as one-fourth of a pie. Then, if she still isn't interested in discussing fractions in the context of division, we might try to think of some other way to bring up the subject, or we might wait a few months. We could talk about dinosaurs and meteors, instead of fractions.

Good luck to you and your child. If we plan our assistance carefully, we can replace authoritative direction with kind support. Children may find the challenge of understanding to be a pleasant diversion, rather than unpleasant work, and they may grow into independent, responsible adults as a result of dealing with our conceptual challenges.

A more standard authoritative approach can be expected to stunt their psychological growth, and result in more insecure, subservient adults. We are masterful when we are careful not to be masters of children, and instead nurture their natural curiosity and their desire to be in charge of their own lives. We needn't disapprove of them, if we can help them understand their errors and learn from them.